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STABILITY IN SYSTEMS WITH AFTEREFFECT WHEN THERE ARE SINGULARITIES IN THE INTEGRAL KERNELS†

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Systems with aftereffect are considered. The state of these systems is described by integrodifferential equations of the Volterra type, which depend on functionals in integral form and, in particular, on analytic functionals which are represented by Frechet series. The integral kernels can allow of singularities of Abel kernel singularities. The total stability (i.e. stability under persistent disturbances) is investigated, and the structure of the general solution is investigated in the neighbourhood of zero for an equation with a holomorphic non-linearity assuming asymptotic stability of the trivial solution of the linearized unperturbed equation. The conditions for instability are given in the critical case of a single zero root, which generalise results obtained previously.

Integrodifferential equations with kernels of the type being considered are used in models of visco-elasticity (in polymer mechanics, for example) and in models of aerodynamics, which take account of the effect on the body of unsteady flow using integral terms.

1. TOTAL STABILITY

We shall consider a system with aftereffect described by an integrodifferential equation of the Volterra type

$$\frac{dx}{dt} = A(t)x + \int_{t_0}^{t} K(t,s)x(s)ds + F(x,y,z,t) + \mu\Phi(\mu,x,y,z,t) \tag{1.1}$$

$$x, y, z \in R^n, x = \operatorname{col}(x_1,\dots,x_n)$$

where

$$y = \int_{t_0}^{t} k(t,s)\varphi(x(s),s)ds \tag{1.2}$$

and $z = \operatorname{col}(z_1, \dots, z_n)$ is an analytic functional which is defined by the Frechet series

$$z(t) = \sum_{k=1}^{\infty} \sum_{j(k)=1}^{n} \int_{t_0}^{t} \dots \int_{t_0}^{t} K^{j(k)}(t,s_1,\dots,s_k)x_{j_1}(s_1)\dots x_{j_k}(s_k)ds_1\dots,ds_k \tag{1.3}$$

The set of indices $j_1, \dots, j_k$ is denoted by $j(k)$.

The $n \times n$ matrix $A(t)$, defined in the set $I = \{t \in R : t \geqslant t_0\}$, has continuous, bounded elements and the $n \times n$ matrix $K(t, s)$ is continuous and is defined in the set $J_1' = \{(t, s) \in R^2 : t_0 \leqslant s < t < +\infty\}$. The continuous $n \times n$ matrix function $k(t, s)$ and the n-vector function $K^{j(k)}(t, s_1, \dots, s_k)$ are defined respectively in the sets J_1' and $J_k' = \{(t, s_1, \dots, s_k) \in R^{k+1} : t_0 \leqslant s_j < t < \infty +, j = 1, \dots, k\}$. The functions $\varphi(x, t)$, $F(x, y, z, t)$, $\Phi(\mu, x, y, z, t)$ are holomorphic, with respect to μ, x, y, z in a certain neighbourhood of zero in the corresponding spaces, continuous, and are bounded with respect to t when $t \in I$ and are such that $\varphi(0, t) \equiv 0$ and the expansion of the function $F(x, y, z, t)$ does not contain terms of lower than the second order.

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In a number of problems in aeromechanics [1, 2] and, also, the mechanics of a deformable body [3, 4], the functionals (1.2) and (1.3) have decreasing integral kernels (of the difference type) which allow of singularities when $t = s$ or $t = s_j$. Here, we shall therefore assume that, in the case of the integral kernels appearing in the representations of (1.2) and (1.3), the following inequalities hold

$$\| k(t,s) \| \leqslant \frac{C\exp[-\beta(t-s)]}{(t-s)^{\rho_0}} \tag{1.4}$$

$$\left\| K^{j(k)}(t,s_1,\ldots,s_k) \right\| \leqslant C\frac{\exp[-\beta_1(t-s_1)-\ldots-\beta_k(t-s_k)]}{[(t-s_1)\ldots(t-s_k)]^{\rho}} \tag{1.5}$$

where $C > 0$, ρ, ρ_0, $\beta > 0$, β_i $(i = 1, \ldots, k)$ are constants, $0 \leqslant \rho_0 < 1$, $0 \leqslant \rho < 1$ and a number β_0 exists, which is independent of k, such that $0 < \beta_0 \leqslant \beta_i$.

In Eq. (1.1), $\mu \geqslant 0$ is a small parameter and the quantity $\mu\Phi(\mu, x, y, z, t)$ is taken as the persistent disturbance.

For Eq. (1.1), we consider the question of the total stability (in the Malkin sense), of the state of the system corresponding to the value $x = 0$. We will also investigate the structure of the general solution of this equation in the neighbourhood of zero, subject to the condition that the solution $x = 0$ of the unperturbed equation (that is, when $\mu\Phi(\mu, x, y, z, t) \equiv 0$) is asymptotically stable. We will use the first Lyapunov method for this purpose and represent the general solution of the Cauchy problem in the form of a power series with respect to the initial values $x_0 = x(t_0) = \mathrm{col}(x_{01}, \ldots, x_{0n})$ and the parameter μ.

The fundamental matrix of the linearized unperturbed equation (1.1), with a lower limit of integration s in the integral term, is denoted by $X(t, s)$ $(X(t, t) = E_n)$. We shall assume that

$$\| X(t,x) \| \leqslant C\exp[-\alpha(t-s)], \quad \alpha = \mathrm{const} > 0 \tag{1.6}$$

Theorem 1. Suppose inequalities (1.4)–(1.6) are satisfied for Eq. (1.1)–(1.3) and that the number $\gamma < \min(\alpha, \beta_0, \beta)$ is chosen.

Then

(1) the general solution of Eq. (1.1)–(1.3) in the neighbourhood of $x = 0$ is represented by the series

$$x(t) = \Gamma(t)\sum_{m=1}^{\infty}\sum_{sl(n)=m} S_1^{l(n)}(t)x_{01}^{l_1}\ldots x_{0n}^{l_n} + \sum_{l_{n+1}=1}^{\infty} S_2^{(l_{n+1})}(t)\mu^{l_{n+1}} +$$
$$+\Gamma(t)\sum_{l_{n+1},m=1}^{\infty}\sum_{sl(n)=m} S_3^{l(n+1)}(t)x_{01}^{l_1}\ldots x_{0n}^{l_n}\mu^{l_{n+1}}, \quad \Gamma(t) = \exp[-\gamma(t-t_0)] \tag{1.7}$$
$$(sl(n) = l_1 + \ldots + l_n)$$

with continuous, bounded coefficients $S_i^{(\cdot)}(t)$ which converge absolutely and uniformly when $\|x_0\| < \delta$, $\mu < \delta$ for any $\delta > 0$;

(2) the point $x = 0$ is totally stable. The *proof* is carried out in a similar way to that described earlier in [6, 7] with the use of the integral equation

$$x(t) = X(t,t_0)x_0 + \int_{t_0}^{t} X(t,s)(F(x(s),y(s),z(s),s) + \mu\Phi(\mu,x(s),y(s),z(s),s))ds \tag{1.8}$$

which is equivalent to Eq. (1.1) with the initial condition x_0. The variables $y(t)$ and $z(t)$ are represented by series similar to (1.7) with the coefficients $S(t)$ with different indices replaced by $P(t)$ and by $Q(t)$ with the same indices, respectively. The above-mentioned coefficients $S(t)$, $P(t)$ and $Q(t)$ are determined successively for increasing m and l_{n+1} on the basis of formulae (1.8), (1.2) and (1.3). For $m = 1$, $l_{n+1} = 1$, for example, we obtain the relations

$$\Gamma(t)\sum_{sl(n)=1} S_1^{l(n)}(t)x_{01}^{l_1}\ldots x_{0n}^{l_n} = X(t,t_0)x_0, \quad S_2^{(1)}(t) = \int_{t_0}^{t} X(t,s)\Phi(0,0,0,0,s)ds \tag{1.9}$$

and the inequalities

$$\|S_1^{l(n)}(t)\| \leqslant C, \ \|S_2^{(1)}(t)\| \leqslant C_1, \ C_1 = \text{const} > 0$$

On the basis of Eqs (1.2), (1.7) and (1.9), we also have

$$\Gamma(t) \sum_{sl(n)=1} P_1^{l(n)}(t) x_{01}^{l_1} \dots x_{0n}^{l_n} + \mu P_2^{(1)}(t) = \int_{t_0}^{t} k(t,s) \varphi'_x(0,s)[X(s,t_0)x_0 + \mu S_2^{(1)}(s)]ds$$

and, consequently, when $sl(n) = 1$, according to the estimate (1.4) and taking into account the fact that $\|\varphi'_x(0,t)\| \leqslant \varphi_0 = \text{const}$, we obtain the estimates for $t \in I$

$$\|P_1^{l(n)}(t)\| \leqslant C^2 \varphi_0 \Gamma^{-1}(t) \int_{t_0}^{t} \frac{\exp[-\beta(t-s)]}{(t-s)^{\rho_0}} \exp[-\alpha(s-t_0)]ds \leqslant C^2 \varphi_0 K^*_{\beta-\gamma,\rho_0}$$

$$\|P_2^1(t)\| \leqslant CC_1 \varphi_0 K^*_{\beta,\rho_0}; \ K^*_{\beta,\rho} = \int_0^\infty \frac{\exp(-\beta\tau)}{\tau^\rho} d\tau$$

In a similar way, for example, in view of relations (1.3), (1.5) and (1,6), we have the limit

$$\|Q_1^{(1)}(t)\| \leqslant C^2 K^*_{\beta_0-\gamma,\rho}$$

for the coefficient $Q_1^{(1)}(t)$ of the expansion for z.

As earlier in [6, 7], the power series

$$u = u(\mu, x_0), \ v = v(\mu, x_0), \ w = w(\mu, x_0)$$

which majorize the expansions for x, y and z respectively, are constructed. To determine them, we have the equations

$$u = C(x_0 + M_1 F^*(u, v, w) + \frac{\mu}{\alpha - \gamma} \Phi^*(\mu, u, v, w)) \tag{1.10}$$

$$v = CK^*_{\beta-\gamma,\rho_0} \varphi^*(u), \ w_i = \frac{CK^*_{\beta_0-\gamma,\rho} u'}{1 - K^*_{\beta_0-\gamma,\rho} u'}, \ u' = u_1 + \dots + u_n \tag{1.11}$$

$$u_i \gg x_i, \ w_i \gg z_i, \ w = \text{col}(w_1, \dots, w_n), \ i = 1, \dots, n$$

where

$$F^*(x,y,z) \gg F(x,y,z,t), \ \Phi^*(\mu,x,y,z) \gg \Phi(\mu,x,y,z,t), \ \varphi^*(x) \gg \varphi^*(x,t)$$

and $1/(\alpha - \gamma)$ can be taken as the constant M_1.

According to the general theory of majorizing equations [8], Eq. (1.10) has a unique positive solution $u = u(\mu, x_0)$ in the form of a converging power series in x_0 and μ which vanishes when $x_0 = 0$, $\mu = 0$. Hence, series (1.7) converges absolutely and uniformly for all $t \in I$ and $\|x_0\| < \delta$, $\mu < \delta$ for a certain $\delta > 0$ which, in turn, implies that the point $x = 0$ is totally stable.

We will now consider Eqs (1.1)–(1.3) in more detail, dropping the requirement that the functions are holomorphic and assuming that a Lyapunov majorant [9] exists for the functions $\varphi(x, t)$, $F(x, y, z, t)$ and for a persistent disturbance $\mu\Phi(x, y, z, t)$ (μ is a small parameter). We shall assume that, in a certain neighbourhood of zero, these functions have $B'(x)$ or $B(x, y, z)$ continuous, bounded first derivatives with respect to x or x, y and z, which are continuous and bounded with respect to $t \in I$, and that $\varphi(0, t) \equiv 0$, $F(0, 0, 0, t) \equiv 0$. The conditions imposed on the functions $A(t)$, $K(t, s)$, $k(t, s)$, $K^{j(k)}(t, s_1, \dots, s_k)$ remain as before; in particular, inequalities (1.4) and (1.5) are satisfied.

Suppose $\varphi^*(x)$, $F^*(x, y, z)$ and $\Phi^*(x, y, z)$ are the Lyapunov majorants for the corresponding functions and, consequently, they are positive and monotonically increasing with respect to x, y and z together with their first derivatives in a certain neighbourhood of zero. We shall assume that the majorants $\varphi^* = \text{col}(\varphi_1^*, \dots, \varphi_n^*)$ and $F^* = \text{col}(F_1^*, \dots, F_n^*)$ satisfy the following conditions for arbitrary ε, which is such that $0 \leqslant \varepsilon \leqslant 1$:

$$\varphi_i^*(\varepsilon u) \leqslant \varepsilon\varphi_i^*(u), \quad u \in B'(u)$$

$$F_i^*(\varepsilon u, \varepsilon v, \varepsilon w) \leqslant \varepsilon^{1+\delta} F_i(u, v, w), \quad \delta > 0, (u, v, w) \in B(u, v, w) \tag{1.12}$$

$$i = 1, \ldots, n$$

We shall also, as earlier, assume that inequality (1.6) is satisfied.
The following theorem holds.

Theorem 2. Suppose Lyapunov majorants $\varphi^*(x)$, $F^*(x,y,z)$ and $\Phi^*(x,y,z)$ exist, which obey inequalities (1.12), in the case of Eq. (1.1)–(1.3) with the functions $\varphi(x, t)$, $F(x, y, z, t)$ and $\Phi(x, y, z, t)$, which satisfy the smoothness and continuity conditions mentioned above. Suppose inequalities (1.4)–(1.6) are satisfied.
Then, the point $x = 0$
(1) is stable under persistent disturbances $\mu\Phi(x, y, z, t)$;
(2) possesses the property of attraction if the condition

$$\|\Phi(0,0,0,t)\| \leqslant C\exp(-\gamma_0 t), \quad \gamma_0 = \text{const} > 0 \tag{1.13}$$

is additionally satisfied.
The *proof* is carried out by the method of successive approximations using Eq. (1.8) and is similar to the proof of Theorem 1 from [10].
If $x_k(t)$, $y_k(t)$, $z_k(t)$ $(k = 1, 2, \ldots)$ are successive approximations for $x(t)$, $y(t)$, $z(t)$

$$x_1(t) = X(t,t_0)x_0 + \mu\int_{t_0}^{t} X(t,s)\Phi(0,0,0,s)ds$$

and

$$u_k \gg x_k(t), \quad v_k \gg y_k(t), \quad w_k \gg z_k(t)$$

then, for the majorants

$$u \gg x(t), \; v \gg y(t), \; w \gg z(t)$$

we have the majorizing equation

$$u = C(x_0 + \frac{1}{\alpha}F^*(u, v, w) + \frac{\mu}{\alpha}\Phi^*(u, v, w)) \tag{1.14}$$

in which expressions (1.11) can be taken for the functions v and w.
Assertion 1 of Theorem 2 follows directly from the existence, in the neighbourhood of the point $u = 0$, of a smooth solution $u = u(\mu, x_0)$ of Eq. (1.14) which increases monotonically with respect to each coordinate and vanishes when $x_0 = 0$, $\mu = 0$.
Assertion 2, which is due to the additional inequality (1.13), follows from the fact that, in this case, the solution can be represented in the form

$$x(t) = \exp(-\gamma t)\tilde{x}(t), \quad \gamma < \min(\alpha, \beta, \beta_0, \gamma_0), \quad \|\tilde{x}(t)\| \leqslant \text{const}, \quad t \in I$$

which can be established in the same way as the analogous property of the solution in Theorem 1 from [10].
As an example, consider the problem [11] of the motion of a rigid body (a wing) when there is an unsteady air flow past it, which, in the unperturbed state, flows round the body at a constant velocity and, in the perturbed state, small fluctuations (gusts) are superimposed on it. These small perturbations are functions of time and, unlike in the case considered previously in [11], they will not be taken as decaying exponentially here. In addition, it can be assumed that the integral kernels $I_{ij}(t)$ and $J_{ij}(t)$ in the representations for the aerodynamic forces and their moments (expression (1.5) in [11]) in the form proposed in [1] are differentiable functions such that their derivatives admit of an estimate of the type of (1.4). Then, on changing from the equations of motion in a form which is unsolved with respect to the derivatives to the standard form of integrodifferential equations, which are solved with respect to the derivatives, we obtain equations of the type of (1.1). In this case, the non-linear terms

of the equations can contain integral terms with kernels containing singularities of the type considered above. Consequently Theorem 1 (or Theorem 2) is applicable, depending on the form of the non-linear terms being considered. The equilibrium position of the wing, which is maintained by viscoelastic springs, the properties of which are maintained the same as in [11], will be totally stable if all the roots of the characteristic equation have negative real parts.

2. STABILITY IN THE CRITICAL CASE OF A SINGLE ZERO ROOT

We will investigate the Lyapunov stability of the motion corresponding to the trivial solution of the equation

$$\frac{dx}{dt} = Ax + \int_{t_0}^{t} K(t-s)x(s)ds + F(x,y,z,t) \tag{2.1}$$

in which A is a constant $n \times n$ matrix and the $n \times n$ matrix $K(t)$ is continuous when $t > 0$ and satisfies an inequality of the type (1.4)

$$\|K(t)\| \leqslant C\frac{\exp(-\beta t)}{t^{\rho_0}}, \quad C>0, \ \beta>0, \ 0 \leqslant \rho_0 < 1 \tag{2.2}$$

The function $F(x, y, z, t)$, which is holomorphic with respect to x, y and z, and a continuous, bounded function with respect to $t \in I$, possesses the same properties as the analogous function in Eq. (1.1) and, moreover, when $t \to +\infty$, the coefficients of the expansion in a power series tend exponentially to constants or they are constant. The variables y and z are given by representations (1.2) and (1.3) in which the integral kernels of the difference type

$$k(t,s) \equiv k_0(t-s), \quad K^{j(k)}(t,s_1,\ldots,s_k) \equiv K_0^{j(k)}(t-s_1,\ldots,t-s_k)$$

are subject to inequalities (1.4) and (1.5).

We will now construct the characteristic equation for Eq. (2.1)

$$\det(\lambda E_n - A - K^*(\lambda)) = 0 \tag{2.3}$$

where $K^*(\lambda)$ is the Laplace transform for $K(t)$.

Suppose that, in the half-plane $\mathrm{Re}\lambda > -\beta$, Eq. (2.3) has a finite number of roots $\lambda_j'(j = 1, \ldots, L, L \geqslant n)$, which have been numbered in the order in which their real parts increase, that is,

$$\mathrm{Re}\,\lambda_1' \leqslant \mathrm{Re}\,\lambda_2' \leqslant \ldots \leqslant \mathrm{Re}\,\lambda_{L-1}' < \lambda_L' = 0 \tag{2.4}$$

We shall assume that the roots $\lambda'_{L-k}(k = 1, \ldots, n-1)$ are simple (there can be complex-conjugate roots among them). For the characteristic exponents we have the relations $\lambda_i = \mathrm{Re}\lambda'_{L+1-i}(i = 1, \ldots, n)$.

The stability in the critical case of a single zero root for Eq. (2.1) was investigated previously in [12–15] in the case when the function F has a simpler structure and the integral kernels do not contain singularities. A technique for determining the Lyapunov constant and a method of proving instability for the example of equations with integral kernels of the exponential-polynomial type were developed in [12–14]. An assertion concerning stability was made in [15] in the case of equations with kernels $K(t) \in C$, without singularities, with a function $F(x, t)$ of the type considered here, and with roots which satisfy condition (2.4).

The result cited below (Theorem 3) extends the corresponding assertion in [15] to the case of integral kernels with singularities of the type of (2.2), (1.4) and (1.5) and with the function $F(x, y, z, t)$ which occurs in Eq. (2.1).

Following the well known procedure [16], we will represent the resolvent of the linearized equation (2.1) in the form

$$R(t) = \sum_{i=L-n+1}^{L} p_i \exp(\lambda_i' t) + R_1(t), \qquad t \in I$$

where p_i is a constant diagonal matrix, the $n \times n$ matrix $R_1(t) \in C^1$ and $\|R_1(t)\| \leqslant C \exp(-\beta^* t)$ $(C, \beta^* = \mathrm{const} > 0)$ for $-\beta < -\beta^* < \lambda_1$. In addition, we shall assume that

$$\| dR_1(t)/dt \| \leqslant C\exp(-\beta' t), \quad \beta' = \text{const} \geqslant \beta^* \tag{2.5}$$

The fundamental matrix of the solutions of the linearized equation (2.1), which is normal in the Lyapunov sense [17, 18], is denoted by $X'(t) = (x'_{ij}(t))$ $(i, j = 1, \dots, n)$.

Suppose $Y'(t) = (y'_{ij}(t))$ is a matrix which is such that $Y'(t)X'(t) = E_n$ and suppose $X'_1(t)$ is a matrix which is obtained from $X'(t)$ by deleting the nth row and the nth column.

Following the approach described earlier in [14, 15], we carry out a transformation which separates out the critical variable and which reduces the linearized equation (2.1) to a differential equation with a constant diagonal matrix. For this purpose, we make the transformation

$$z' = \text{col}(z_1, \dots, z_{n-1}) = \exp(\Lambda' t) Y_1'(t) x'$$
$$x' = \text{col}(x_1, \dots, x_{n-1}), \quad \Lambda_1' = \text{diag}(\lambda'_{L-n+1}, \dots, \lambda'_{L-1})$$

and introduce the critical variable

$$z_n = \sum_{j=1}^{n} y'_{nj}(t) x_j$$

At the same time, it is assumed that the following conditions, which permit the transformations under consideration to belong to the class of Lyapunov transformations, are satisfied

$$\begin{aligned} &\| \exp(-\sum_{j=1}^{n-1} \lambda_j t) \det X'(t) \| \geqslant d' > 0, \qquad t \in I \\ &\| y'_{nn}(t) \| \geqslant \delta' > 0, \qquad d', \delta', h' = \text{const} \\ &\| \exp(-\sum_{j=1}^{n-1} \lambda_j t) \det X_1'(t) \| \geqslant h' > 0 \end{aligned} \tag{2.6}$$

We shall use the following definitions.

We shall say that the function $f(t) \in e_1(\alpha)$, if, when $t \in I$, the estimate

$$\| f(t) \| \leqslant C\exp(\alpha t), \quad C > 0, \quad \alpha = \text{const}$$

holds.

We shall also say that the function $\varphi(t, s) \in e_2'(\gamma, \alpha)$, if, when $t_0 \leqslant s < t < +\infty$, the inequality

$$\| \varphi(t,s) \| \leqslant C\frac{\exp[\alpha(t-s)]}{(t-s)^\gamma}, \quad C > 0, \quad 1 > \gamma \geqslant 0, \quad \alpha = \text{const}$$

holds.

Similarly, the function $\psi(t, s_1, s_2, \dots, s_k) \in e'_{k+1}(\gamma, \alpha_1, \dots, \alpha_k)(\alpha_j = \text{const})$, if, in J'_k

$$\| \psi(t, s_1, \dots, s_k) \| \leqslant C\frac{\exp[\alpha_1(t-s_1) + \dots + \alpha_k(t-s_k)]}{[(t-s_1)\dots(t-s_k)]^\gamma}$$

If the last inequality holds when $\gamma = 0$ for $t_0 \leqslant s_j \leqslant t < +\infty$ $(j = 1, \dots, k)$, we shall assume that $y(t, s_1, s_2, \dots, s_k) \in e_{k+1}(\alpha_1, \dots, \alpha_k)$. If, in this case, $\alpha_1 = \dots = \alpha_k = \alpha$, then we shall also denote $e'_{k+1}(\gamma, \alpha_1, \dots, \alpha_k)$ and $e_{k+1}(\alpha_1, \dots, \alpha_k)$ by $e'_{k+1}(\gamma, \alpha)$ and $e_{k+1}(\alpha)$, respectively.

All of the transformations which have been performed, which enable one to separate out the Lyapunov constant and to prove instability, must retain the property of all the integral kernels to belong to the class $e_{k+1}(-\alpha_1, \dots, -\alpha_k)$ or $e'_{k+1}(\gamma, -\alpha_1, \dots, -\alpha_k)$ $(\alpha_j > 0)$ and the property of all the coefficients $\varphi(t)$ of terms not containing integrals to decrease exponentially when $t \to +\infty$, that is

$$\varphi(t) = \varphi_0 + \varphi_1(t), \quad \varphi_0 = \text{const}, \quad \varphi_1(t) \in e_1(-\alpha), \quad \alpha > 0 \tag{2.7}$$

We will estimate certain coefficients with integral terms which arise when carrying out the transformations. For instance, for $K(t-s) \in e_2'(\gamma, -\beta)$ $(\beta > 0)$, we obtain

$$\int_{t_0}^{t} K(t-s) ds = \int_{t_0}^{\infty} K(s) ds - \int_{t}^{\infty} K(s) ds = k_0 + k_1(t)$$

that is, we have a function of the type (2.7) with $k_1(t) \in e_1(-\beta)$.

Suppose $K(t, s) \in e_2'(\gamma, -\beta)$ and $f(t) \in e_1(-\alpha')$ when $\beta > 0$, $\alpha' > 0$. Then

$$\left\| \int_{t_0}^{t} K(t,s) f(s) ds \right\| \leqslant C \exp(-\alpha' t) \int_{t_0}^{t} \frac{\exp[-(\beta - \alpha')(t-s)]}{(t-s)^{\gamma}} ds \in e_1(-\delta + \varepsilon) \tag{2.8}$$

where $\delta = \min(\alpha', \beta)$ and $\varepsilon > 0$ is a certain small number such that $\delta + \varepsilon < 0$.

We will now consider a transformation which is similar to that carried out in [14, Section 3] and enables us to eliminate integral terms that are linear with respect to z_n from the subsystem for the non-critical variables. Retaining the notation from [14], we estimate, for example, the function introduced there

$$h_1(t, \tau) = \int_{t_0}^{\tau} k(t,s) ds$$

where the integral kernel, if account is taken of (2.2) and the formula for $k(t, s)$, is such that $k(t, s) \in e_2'(\rho_0, -\alpha)$ for a certain $\alpha > 0$. We have

$$\| h_1(t, \tau) \| \leqslant C \int_{t-\tau}^{t-t_0} \frac{\exp(-\alpha s)}{s^{\rho_0}} ds < C \int_{t-\tau}^{\infty} \frac{\exp(-\alpha s)}{s^{\rho_0}} ds \tag{2.9}$$

It follows from (2.5) that $h_1(t, \tau) \in e_2(-\alpha)$.

Actually, on extending the definition of the function $h_1(t, \tau)$ with respect to continuity when $t = \tau$ and estimating the integral on the right-hand side of inequality (2.9) (a bounded function when $0 \leqslant t - \tau < +\infty$), we obtain for $t - \tau \geqslant 1$

$$\int_{t-\tau}^{\infty} \frac{\exp(-\alpha s)}{s^{\rho_0}} ds < \int_{t-\tau}^{\infty} \exp(-\alpha s) ds$$

Consequently, we have

$$\| h_1(t, \tau) \| \leqslant C' \exp[-\alpha(t-\tau)], \quad 0 \leqslant \tau \leqslant t < +\infty, \quad C' = \text{const} > 0$$

The estimates and the form of the decay of the functions $h_2(t, s)$, $R_i(t, s)$ ($i = 1, 2, 3$), introduced in [14], are also retained. We will estimate, for example, the integral of the form

$$I_1(t, \tau) = \int_{\tau}^{t} h(s, \tau) h'(t, s) ds, \quad h \in e_2'(\gamma_1, -\alpha_1), \quad h' \in e_2'(\gamma_2, -\alpha_2), \quad \alpha_1 > 0, \quad \alpha_2 > 0$$

which appears during the course of the transformations.

We have

$$| I_1(t, \tau) | \leqslant C' \int_{\tau}^{t} \frac{\exp[-\alpha_1(s-\tau)]}{(s-\tau)^{\gamma_1}} \frac{\exp[-\alpha_2(t-s)]}{(t-s)^{\gamma_2}} ds <$$

$$< C' \int_{\tau}^{t} \frac{\exp[-\alpha_1(s-\tau)]}{(s-\tau)^{\gamma_1}} ds \int_{\tau}^{t} \frac{\exp[-\alpha_2(t-s)]}{(t-s)^{\gamma_2}} ds =$$

$$= C' \int_{0}^{t-\tau} \frac{\exp(-\alpha_1 s)}{s^{\gamma_1}} ds \int_{0}^{t-\tau} \frac{\exp(-\alpha_2 s)}{s^{\gamma_2}} ds, \quad 0 \leqslant \tau < s < t < +\infty \tag{2.10}$$

and, hence, $I_1(t, \tau) \in e_2(-\alpha)$ for $\alpha = \min(\alpha_1, \alpha_2)$.

As a result, a system of equations

$$\frac{dz_n}{dt} = Z_n(z', z_n, t) \tag{2.11}$$

$$\frac{dz'}{dt} = \Lambda_1' z' + \int_{t_0}^{t} R_3(t,\tau) Z_n(z', z_n, \tau) d\tau - \int_{t_0}^{t} R_2(t,\tau) \Phi(z', z_n, \tau) d\tau + \Phi'(z', z_n, t) \tag{2.12}$$

is obtained at this stage of the transformations, where

$$R_3(t,\tau) = R_1(t,\tau) + \int_{\tau}^{t} h_1(s,\tau) R_2(t,s) ds \in e_2(-\alpha)$$

for a certain $\alpha > 0$, the expansions for Z_n, Φ begin with the quadratic terms and the expansion for Φ' also contains a term which is linear in z_n. The order of a certain term occurring in $\varphi(z, z_n, t)$ is determined by the power of the parameter ε in this term in the expansion with respect to ε for $\varphi(\varepsilon z, \varepsilon z_n, t)$ with the replacement $z \to \varepsilon z$, $z_n \to \varepsilon z_n$ (in the integrands as well).

Equations (2.11) and (2.12) contain integral terms with integral kernels which only satisfy estimates of the type (1.4) and (1.5) or which belong to the class $e_k(-\alpha)$ for certain k and $\alpha > 0$. In determining the Lyapunov constant in Eq. (2.11), the integral terms of order k, which are solely dependent on the critical variable z_n, are transformed by integration by parts in order to separate out the non-integral term of the same order. For example, if $\varkappa(t-s) \in e_2'(\gamma, -\alpha)$, $\tilde{\varphi}(t) \in e_1(-\alpha_0)$ for $\alpha > 0$, $\alpha_0 > 0$, we have

$$\int_{t_0}^{t} \varkappa(t-s)\tilde{\varphi}(s) z_n^k(s) ds = z_n^k I'(t,t) - \int_{t_0}^{t} I'(t,\tau) k z_n^{k-1}(\tau) Z_n(z'(\tau), z_n(\tau), \tau) d\tau$$

$$I'(t,\tau) = \int_{t_0}^{\tau} \varkappa(t-s)\tilde{\varphi}(s) ds$$

where, according to inequality (2.8), $I'(t,t) \in e_1(-\gamma')$ for a certain $\gamma' > 0$. By analogy with inequality (2.8), we have $I'(t,\tau) \in e_2(-\gamma'')$ for a certain $\gamma'' > 0$.

The transformation and estimation of the terms of the Frechet series, which depend solely on the critical variables, are carried out in a similar manner. In particular, for the quadratic term of the Frechet series with integral kernel $K^{n,n}(t-s_1, t-s_2) \in e_3'(\gamma, -\beta_0)$ $(\beta_0 > 0)$, we obtain

$$\int_{t_0}^{t}\int_{t_0}^{t} K^{n,n}(t-s_1, t-s_2) z_n(s_1) z_n(s_2) ds_1 ds_2 = \int_{t_0}^{t} [z_n(t) \int_{t_0}^{t} K^{n,n}(t-s_1, t-s_2) ds_1 -$$

$$- \int_{t_0}^{t}\int_{t_0}^{s_1} K^{n,n}(t-\tau, t-s_2) d\tau Z_n(z'(s_1), z_n(s_1), s_1) ds_1] z_n(s_2) ds_2 = z_n^2(t) k^{n,n}(t) + \ldots$$

where the dots denotes terms of higher than the second order. The coefficient $k^{n,n}(t)$ is given by the expression

$$k^{n,n}(t) = \int_{t_0}^{t}\int_{t_0}^{t} K^{n,n}(s_1, s_2) ds_1 ds_2 = k_0^{n,n} + k_1^{n,n}(t); \quad k_0^{n,n} = \int_{t_0}^{\infty}\int_{t_0}^{\infty} K^{n,n}(s_1, s_2) ds_1 ds_2$$

where $k_0^{n,n}$ is a constant and $k_1^{n,n}(t) \in e_1(-\alpha)$ for a certain $\alpha > 0$.

The transformation of subsystem (2.12) for the non-critical variables reduces to eliminating terms from the right-hand side which depend solely on z_n up to a certain power k_1 and integral terms which are linear with respect to z_1 and contain z_n in powers which do not exceed a certain number k_2. For example, to determine the constant g_2 from Eq. (2.11) using the substitution

$$u = z' + u_1'(t) z_n + u_2'(t) z_n^2 + \int_{t_0}^{t} M_1(t,s) z'(s) z_n(s) ds \tag{2.13}$$

linear and quadratic terms in z_n are eliminated and, also, the quadratic integral term containing $z'z_n$. In (2.13), $u_1'(t)$, $u_2'(t)$ are continuous and bounded functions when $t \in I$ and the function $M_1(t,s)$ is continuous when $(t,s) \in J_1'$. These functions are determined in the same way as in [14] and, in particular, we have the expression

$$M_1(t,s) = -\exp(\Lambda_1' t) \int_{s}^{t} \exp(-\Lambda_1' \tau) M_1'(\tau, s) d\tau$$

where $M_1'(\tau, s)$ is the specified kernel of the integral term which is subject to elimination. At the same time, $M_1'(\tau, s) \in e_2'(\gamma, -\alpha)$ $(\alpha > 0)$ and, consequently, according to inequality (2.10), $M_1(t, s) \in e_2'(-\alpha')$ for a certain $\alpha' > 0$.

In the general case, when determining the constant g_{2m+1} $(g_k = 0, k = 2, 3, \dots, 2m)$, for example, a transformation of the type of (2.13) is carried out in which there is a polynomial of degree $4m$ in z_n and an integral term which is linear in z_1 and is a polynomial of degree $2m + 1$ in z_n. The coefficients $u_i'(t)$ and the kernels $M_p(t, s)$ of this transformation are found in the same way as in (2.13) and are such that $M_p(t, s) \in e_2'(\gamma, -\alpha)$ or $M_p(t, s) \in e_2(-\alpha)$ $(\alpha > 0)$ and $u_i'(t)$ is of the type (2.7). Next, the Lyapunov constant g_p is separated out in the equation for the critical variable by the standard procedure [13, 14] and this equation for the new critical variable u_n takes the form

$$\frac{du_n}{dt} = g_p u_n^p + U_n^{(2)}(u, u_n, t) + U_n^{(p+1)}(u, u_n, t) \tag{2.14}$$

where $U_n^{(2)}$, $U_n^{(p+1)}$ are integral operators such that $U_n^{(2)}(0, u_n, t) \equiv 0$ and $U_n^{(2)}(\varepsilon u, \varepsilon u_n, t)$ is a polynomial in ε of degree p without free and linear terms and the expansion with respect to ε for $U_n^{(p+1)}(\varepsilon u, \varepsilon u_n, t)$ starts from the term containing ε^{p+1}.

Next, as previously in [13, 14], if $p = 2m$ and $g_p \neq 0$ or $p = 2m + 1$ and $g_p > 0$, a sector is constructed in which the trajectory departs from the point $x = 0$ and the instability of the zero solution is established using Chetayev's theorem on instability.

The following result therefore holds.

Theorem 3. Suppose the characteristic equation (2.3) for Eq. (2.1), (1.2), (1.3), (2.2) has a finite number of roots λ_j' $(j = 1, \dots, L)$ in the complex half-plane $\operatorname{Re}\lambda > -\beta$, $\lambda_L' = 0$ and inequalities (2.4) hold. Also, suppose conditions (1.4), (1.5), (2.5) and (2.6) are satisfied and that the constant $g_p \neq 0$ $(g_s = 0, s = 2, \dots, p - 1)$ when p is even or $g_p > 0$ when p is odd.

Then, the trivial solution of Eq. (2.1), (1.2), (1.3) is unstable.

Returning to the problem of a rigid body in an unsteady flow, considered in Section 1, we note that the results previously obtained [11] on the instability of the equilibrium of a body in the critical case of a zero root can be naturally extended on the basis of Theorem 3 to the case of integral kernels admitting of the estimates (1.4), (1.5) and (2.2).

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